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1970 J. Phys. A: Gen. Phys. 3 501

(http://iopscience.iop.org/0022-3689/3/5/006)

J. Phys. A: Gen. Phys., 1970, Vol. 3. Printed in Great Britain

Null electromagnetic fields in general relativity

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MS. received 16*th March* 1970

Abstract. We present here a class of solutions of the Einstein–Maxwell equations corresponding to stationary null electromagnetic fields in otherwise empty space.

1. Introduction

Many attempts have so far been made in the direction of finding exact solutions of the Einstein–Maxwell equations corresponding to null electromagnetic field. In recent years Ozsvath (1966) and Dutta and Raychaudhuri (1968) have investigated a special kind of null electromagnetic field which is stationary in the sense that the spacetime in either of them admits a group of motion with a time-like generator. Ozsvath's solutions correspond to dust-radiation universes in which the stress–energy tensor is partly due to incoherent dust and partly due to null electromagnetic field, while in the solutions presented by Dutta and Raychaudhuri, however, the electromagnetic field is primarily assumed to be independent of time representing purely static electric and magnetic fields being superimposed upon each other orthogonally.

In the present paper we have started with a line element (Van Stockum 1937) used by Dutta and Raychaudhuri in their work and we have found more general solutions corresponding to stationary null electromagnetic field in matter-free space in which the null propagation vector satisfies Robinson's condition (1959). One of the metric components in our solutions, however, remains arbitrary. In the last part we have discussed two special cases, one of which represents the purely static electromagnetic null field, already dealt with by Dutta and Raychaudhuri.

2. Null field solutions

We start with the line element of the form (Van Stockum 1937)

$$ds^2 = f\,dt^2 - e^{2\psi}(dx^2+dy^2) - l\,dz^2 + 2m\,dz\,dt \tag{1}$$

where f, l, m and ψ are functions of x alone. Now, as is well known, the null electromagnetic field tensor ($F_{\alpha\beta}F^{\alpha\beta} = {}^*F_{\alpha\beta}F^{\alpha\beta} = 0$) can be written in the form

$$F_{\alpha\beta} = (K_\alpha\xi_\beta - K_\beta\xi_\alpha) \tag{2}$$

where the null vector K^α is the propagation vector and ξ^α is the space-like vector orthogonal to the propagation vector. Again without loss of generality ξ^α may be chosen to be a unit vector so that the field equation for matter-free pure radiation is

$$R_{\alpha\beta} = -2K_\alpha K_\beta. \tag{3}$$

The relation (3) with the line element (1), however, requires $K^1 = K^2 = 0$. K^3 and K^4 are thus the only non-vanishing contravariant components of K^α. Again in view of the fact that K^α is a null vector the relation (3) requires

$$R_3{}^3 + R_4{}^4 = 0$$

and thus allows one to use Weyl-like canonical coordinates (Van Stockum) so that

$$fl+m^2 = x^2. \tag{4}$$

Now from field equations one can find the following relations:

$$x\psi_{11}-\psi_1-\frac{1}{2x}(f_1l_1+m_1{}^2) = 0 \tag{5a}$$

$$x\psi_{11}+\psi_1 = 0 \tag{5b}$$

$$\frac{1}{2(-g)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{fl_1+mm_1}{x}\right) = 2K^3K_3 \tag{5c}$$

$$\frac{1}{2(-g)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{lf_1+mm_1}{x}\right) = 2K^4K_4 \tag{5d}$$

$$\frac{1}{2(-g)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{mf_1-fm_1}{x}\right) = 2K^3K_4 \tag{5e}$$

and

$$\frac{1}{2(-g)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{lm_1-ml_1}{x}\right) = 2K^4K_3. \tag{5f}$$

The equation (5*b*) after straightforward calculation leads to the relation

$$\mathrm{e}^{\psi} = x^{\alpha} \tag{6}$$

α being the integration constant.

Again from (5*a*):

$$f_1l_1+m_1{}^2 = -4\alpha. \tag{7}$$

Combining (4) and (7) and integrating, together with the assumption $4\alpha = -1$, one finally obtains

$$m = al-x \tag{8a}$$

$$f = (2ax-a^2l) \tag{8b}$$

$$\mathrm{e}^{\psi} = x^{-1/4}. \tag{8c}$$

Equations (8*a*) and (8*b*) may alternatively be written as

$$m = al+x \tag{9a}$$

and

$$f = -(2ax+a^2l). \tag{9b}$$

They are, however, equivalent, corresponding to the transformation $x \to -x$. Using now the relations (8*a*), (8*b*), (8*c*) in (5*c*) and (5*e*), and remembering that K^α is a null vector, one gets

$$\left.\begin{aligned} K_4 &= -aK_3 \\ K^4 &= \frac{1}{a}K^3 \end{aligned}\right\} \tag{10}$$

which can as well be obtained from (5*d*) and (5*f*).

It is not difficult to show that the solutions (8*a*), (8*b*) and (8*c*) completely satisfy the field equations irrespective of the value of l and that the null propagation vector K^α satisfies, in view of the relation (10), the condition

$$K^{\alpha}_{;\alpha} = K^{\alpha}_{;\beta}K^{\beta} = K_{(\alpha;\beta)}K^{\alpha;\beta} = 0 \tag{11}$$

which is fully consistent with Robinson's (1959) condition for the field to be a vacuum null electromagnetic field. The interesting feature of the solutions presented above is that one of the metric components (namely l) remains arbitrary and one can obtain special solutions of interest by suitably choosing its value.

3. Two special cases

3.1. *Static null electromagnetic field*

For a purely static electromagnetic field the $F^{\mu\nu}$ tensor is independent of time. Assuming that F^{31} and F^{41} are the only non-vanishing contravariant components of the electromagnetic field tensors which are functions of x alone, one can write

$$\left.\begin{aligned} F^{31} &= \frac{A}{(-g)^{1/2}} \\ F^{41} &= \frac{B}{(-g)^{1/2}} \end{aligned}\right\}. \tag{12}$$

Since ξ^{μ} in (2) is a unit space-like vector ($\xi^{\mu}\xi_{\mu} = -1$), the relation (12) leads to

and
$$\left.\begin{aligned} K^3 = aK^4 &= A/x^{3/4} \\ A &= aB \end{aligned}\right\}. \tag{13}$$

Now in view of relations (8*a*), (8*b*), (8*c*) and (13) the equation (5*c*) leads to

$$\frac{l}{x} = -(4B^2x + C\ln x + D). \tag{14}$$

The same result may also be obtained if one uses relations (9*a*) and (9*b*) instead of (8*a*) and (8*b*). Here B, C, D are integration constants. Now the solutions (14) together with (9*a*) and (9*b*) are exactly identical with those obtained by Dutta and Raychaudhuri.

3.2. $g_{33} = 0$

Let the Killing equation

$$K_{\mu;\nu} + K_{\nu;\mu} = 0 \tag{15}$$

be satisfield, that is, the spacetime admits a null translation. One can now choose the coordinate system so that the motion is a translation along the null coordinate z thus $K^{\mu} = \delta^{\mu}_{3}$ and

$$l = 0. \tag{16}$$

From (4) it then follows that

$$m = \pm x. \tag{17}$$

The only non-zero component of R^{μ} is R^3_4 and hence by solving the equation (5*e*) one gets

$$f = \frac{16}{25}x^{7/2} + Bx\ln x + Cx. \tag{18}$$

Where B, C are integration constants. The solutions (16), (17) and (18) together with $e^{\psi} = x^{-1/4}$ satisfy all the field equations. It is, however, interesting to note that in this case the antisymmetric tensor $K_{[\mu;\nu]} \neq 0$.

It turns out from the above discussions that the null field presented here is due to divergence-free electromagnetic waves. Moreover, the rays are geodetic and shear-free in agreement with Robinson's conditions. The interesting feature of these solutions is that they exhibit, on the one hand, the dynamic characteristics in which there is a flux of energy flow without any accumulation of it anywhere in the course of time, so as not to disturb the stationary nature of the metric; while on the other hand the solutions also include the case of static electromagnetic null field, made up of purely static electric and magnetic fields, being superimposed upon each other orthogonally.

Acknowledgments

The author is grateful to Professor A. K. Raychaudhuri of Presidency College, Calcutta, for useful suggestions.

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